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2007 J. Phys. A: Math. Theor. 40 9387

(http://iopscience.iop.org/1751-8121/40/31/015)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 9387–9397 doi:10.1088/1751-8113/40/31/015

Twisted quantum $\mathbb{Z}_n$ modular data and braided subfactors

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Received 17 May 2007, in final form 12 June 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9387

Abstract

The modular data from twisted $[k] \in \mathrm{H}^3(G, \mathbb{T})$ quantum double of finite groups G are studied, with $G = \mathbb{Z}_n$ cyclic groups. It is proved that the $[k]$ and $[-k]$ modular data are the same up to relabelling of the primary fields and complex conjugation of the underlying representation of the modular group $SL(2, \mathbb{Z})$. Then we produce some lower bounds for the number of modular invariants of these models, and complete the study for the cases $G = \mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_4$ at all twists, proving in particular that all their modular invariants are produced by braided subfactors.

PACS numbers: 02.30.Sa, 11.25.Hf, 02.40.Gh, 02.20.Uw
Mathematics Subject Classification: 46L60, 81T40, 46L37, 81T05

1. Motivation and introduction

We use the braided subfactor framework developed in [2–5, 13] to further the study of the modular data arising from the quantum double of finite groups G, possibly with twists or levels $[k] \in \mathrm{H}^3(G, \mathbb{T})$. These models first appeared in [7–9] as holomorfic orbifolds models and more recently in [6, 13, 28]. Perhaps the most physical incarnation of this modular data is in the (2+1)-dimensional quantum field theories [1], where a continuous gauge group has been spontaneously broken to a finite group.

We borrow techniques from the noncommutative setting of subfactor theory and begin a more exhaustive study of the modular invariants associated with these *twisted* models, starting with cyclic groups $G = \mathbb{Z}_n$.

More generally, a prominent problem in rational conformal field theory (RCFT) is the classification of modular invariant partition functions $Z(\tau) = \sum Z_{\lambda,\mu}\chi_\lambda(\tau)\chi^*_\mu(\tau)$, where $\chi_\lambda(\tau) = \mathrm{Tr}_{H_\lambda}(\mathrm{e}^{2\pi \mathrm{i}\tau(L_0 - c/24)})$ is the trace in the irreducible representation λ (primary field) of the chiral algebra, with conformal Hamiltonian L_0, $\mathrm{Im}(\tau) > 0$ and c is the central charge.

This problem has been solved for a few models, although its mathematical formulation is very simple in terms of the following modular data. For a given finite-dimensional representation of the modular group $SL(2,\mathbb{Z})$, let $S = [S_{\lambda,\mu}]$ and $T = [T_{\lambda,\mu}]$ denote the matrices representing the images of the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, respectively. We further suppose that T is diagonal, S is symmetric, S^2 a permutation matrix and $S_{\lambda,0} \geqslant S_{0,0} > 0$ where '0' is a distinguished primary field (the vacuum). Then a coupling matrix $Z = [Z_{\mu\lambda}]$ that commutes with S and T subject to the constraints

$$Z_{\lambda,\mu} = 0, 1, 2, 3, \ldots \qquad \text{and} \qquad Z_{00} = 1$$

is called a modular invariant. These constraints reflect the physical background of the problem. The condition $Z_{00} = 1$ reflects the uniqueness of the vacuum. In the partition $Z(\tau)$ formulation, the modular invariance can be rephrased as follows:

$$Z\left(\frac{a\tau + b}{c\tau + d}\right) = Z(\tau) \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Note that the identity $Z = \mathrm{id}$ and the charge conjugation $C := S^2$ are always modular invariants, possibly the same.

In our framework, each primary field has a concrete meaning of an endomorphism $\lambda \in \mathrm{End}(N)$ in a fixed type III von Neumann factor N and thus the set of primary fields will be represented by a (finite) set of endomorphisms ${}_N\mathcal{X}_N$ on N which we call a system of endomorphisms and are moreover assumed to be nondegenerately braided [4, 5]. The vacuum '0' is represented by the identity endomorphism of N. Hence, we can yield such modular matrices S and T [4, 5], a representation of the modular group $SL(2,\mathbb{Z})$ satisfying the above constraints. Therefore, we can produce a list of modular invariants, which *a priori* we do not know whether they are physically meaningful. Given a braided subfactor $N \subset M$ (i.e. the inclusion morphism $\iota : N \to M$ is such that the endomorphism $\theta := \bar{\iota}\iota$ of N decomposes as a sum of endomorphisms from ${}_N\mathcal{X}_N$) we can apply the Longo–Rehren induction [23, 32] to extend each $\lambda \in {}_N\mathcal{X}_N$ to two morphisms $\alpha_\lambda^{\pm} \in \mathrm{End}(M)$ since we always have two choices for the braiding. The morphism $\bar{\iota} : M \to N$ is the conjugate morphism of ι. Now if we consider the intertwiner space

$$\mathrm{Hom}\big(\alpha_\lambda^+, \alpha_\mu^-\big) = \big\{x \in M : x\alpha_\lambda^+(y) = \alpha_\mu^-(y)x, \text{ for all } y \in M\big\}, \tag{1}$$

and its dimension $Z_{\lambda\mu} = \dim \mathrm{Hom}\big(\alpha_\lambda^+, \alpha_\mu^-\big)$, then the matrix $Z_{N\subset M} = [Z_{\lambda\mu}]$ is indeed a modular invariant [4]. (In general for morphisms σ, ρ from A to B we set $\mathrm{Hom}(\sigma,\rho) = \{b \in B : b\sigma(a) = \rho(a)b, \text{ for all } a \in A\}$ and $\langle\sigma,\rho\rangle = \dim \mathrm{Hom}(\sigma,\rho)$.) Given the list of modular invariants, one of the main interesting tasks is to decide which ones can be realized through braided subfactors (the so-called *sufferable* modular invariants). There is a considerable evidence that the sufferable modular invariants are precisely those of physical interest, see, e.g., [13, p 312]. See [13, 14] for modular invariants that cannot appear from subfactors and thus physically unhealthy. Of course, we may have different subfactors producing the same modular invariant.

Let $\iota : N \subset M$ be a subfactor. Since $\bar{\iota}\iota, \iota\bar{\iota}$ contain the identity $\mathrm{id}_M, \mathrm{id}_N$, respectively, there are intertwining isometries, v and w_1, in $\mathrm{Hom}(\mathrm{id}_N, \bar{\iota}\iota)$, $\mathrm{Hom}(\mathrm{id}_M, \iota\bar{\iota})$, respectively [20, 22]. Then $w = \bar{\iota}(w_1)$ is an isometry in $\mathrm{Hom}(\theta, \theta^2)$ where $\theta = \bar{\iota}\iota$ is the dual canonical endomorphism which satisfy [22]

$$w^*\theta(w) = ww^*, \qquad w^2 = \theta(w)w, \qquad v^*w = w^*\theta(v) = 1/d \tag{2}$$

with $d = \dim(\iota)$, thus $d^2 = [M : N]$ is the Jones index [21]. The system $\Theta = (\theta, v, w)$ is called a Q-system by Longo [22], and conditions in (2) precisely characterize which endomorphisms can arise as dual canonical endomorphisms for $N \subset M$. Conversely, given a Q-system we can always produce an inclusion $N \subset M$, see [22]. However, a dual endomorphism θ does not determine the subfactor uniquely up to conjugacy. This is an H^2-cohomological obstruction that has been studied in [18] and [13, proposition 3.2 and remark 3.3].

In modular tensor category setting, the notion of Q-system is named Frobenius algebra, see, e.g., [14, 16]. Namely, the above relations (2) mean that a Q-system is a Frobenius algebra $A = (\theta, m, e, \Delta, \epsilon)$, see [25], where θ is an object of $\mathcal{C}$ (an integral sum of simple objects in a modular tensor category $\mathcal{C}$), $e \in \mathrm{Hom}(\mathbf{1}, \theta)$, $m \in \mathrm{Hom}(\theta \otimes \theta, \theta)$, $\epsilon \in \mathrm{Hom}(\theta, \mathbf{1})$, $\Delta \in \mathrm{Hom}(\theta, \theta \otimes \theta)$ such that (θ, m, e) is an algebra, $(\theta, \Delta, \epsilon)$ is a co-algebra with the algebraic and co-algebraic structures related by

$$(m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ m = (\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A). \tag{3}$$

The intertwiners w and w^* are translated into Δ and m, respectively, and the isometry v and v^* are replaced by e and ϵ, respectively. See [14] for full details.

The set of modular invariants that can be realized by subfactors enjoys a very rich structure: for example, if Z_a and Z_b are produced by dual endomorphisms θ_a and θ_b, respectively, then the matrix product $Z_a Z_b$ is produced by the composition $\theta_a \theta_b$ with inclusion $N \subset M_{ab}$. Of course, $Z_a Z_b$ is a modular invariant except that its (0, 0)-entry may no longer be normalized to be 1, see [13]. In fact, $[Z_a Z_b]_{00} = \dim(M_{ab} \cap M'_{ab})$, where M'_{ab} denotes the comutant of M_{ab}, see [13]. Moreover, we obtain a (possibly noncommutative) fusion structure [13]:

$$Z_a Z_b = \sum_c m^c_{ab} Z_c, \quad \text{with} \quad m^c_{ab} \in \mathbb{N}_0, \quad Z_c \text{ sufferable and normalized.} \tag{4}$$

This fusion structure is quite useful specially when we deal with a large number of modular invariants, as we see in this paper.

Of course, in general when we have a subfactor and therefore a dual endomorphism θ it is difficult to use (1) to pin down the associated modular invariant $Z_{N\subset M}$. Nevertheless, we may start by computing its trace $\mathrm{Tr}(Z_{N\subset M})$. This can be done by counting the irreducible $N - M$ morphisms from the decompositions $\iota\lambda$ with $\lambda \in {}_N\mathcal{X}_N$, see [5]. This is the ${}_N\mathcal{X}_M$ system and to help find the common part of the irreducible decompositions of $\iota\lambda$ and $\iota\mu$ we appeal to Frobenius reciprocity [4, 20] and get

$$\langle \iota\lambda, \iota\mu \rangle = \langle \bar{\iota}\iota\lambda, \mu \rangle = \langle \theta\lambda, \mu \rangle = \langle \theta, \bar{\lambda}\mu \rangle, \tag{5}$$

and $[\bar{\lambda}\mu] = \sum N^{\nu}_{\bar{\lambda}\mu}[\nu]$ where $N^{\nu}_{\bar{\lambda}\mu} \in \mathbb{N}_0$ are the famous Verlinde fusion numbers which are derived from the S matrix as follows [31]:

$$\sum_{\xi \in {}_N\mathcal{X}_N} \frac{S_{\lambda,\xi} S_{\mu,\xi} S^*_{\nu,\xi}}{S_{\xi,0}} = N^{\nu}_{\lambda,\mu}. \tag{6}$$

In the subfactor framework, we further have $N^{\nu}_{\lambda,\mu} = \langle \lambda\mu, \nu \rangle$. If Z is the modular invariant yielded by $N \subset M$ then the following 'curious' identity holds

$$\bigoplus_{a \in {}_N\mathcal{X}_M} [\bar{a}a] = \bigoplus_{\lambda,\mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu}[\lambda\bar{\mu}] \tag{7}$$

which will be useful in the following.

We can find modular data in a wide variety of contexts, including the Weiss–Zumino–Witten (WZW) models [33] (notably the Ising model see, e.g., [14, section 4.2]), the Drinfeld quantum double (or the operator algebras Ocneanu's quantum double analogue [26, 19]), notably the quantum double of finite groups where the primary fields λ are labelled by pair (a, π) with a running in a set of conjugacy class representatives of the group G and π are the irreducible representations of the centralizer $C_G(a) = \{g \in G : ga = ag\}$. For example, for the cyclic group $\mathbb{Z}_n$ we have n conjugacy classes and the centralizer of every class coincides with $\mathbb{Z}_n$, thus altogether we have $\mathbb{Z}_n \times \mathbb{Z}_n$ primary fields.

One way to generalize the (quantum double) finite group data is by introducing twistings. This twisting has a cohomological origin, as in the theory of WZW G models for compact Lie groups G, where infinitely many possible twists are labelled by the levels k, which are integers since $\mathrm{H}^3(G, \mathbb{T}) = \mathbb{Z}$. The level 0 WZW model is trivial.

In contrast with the WZW models, the twisting (the elements of the finite Abelian group $\mathrm{H}^3(G, \mathbb{T})$) of finite groups offers a finite number of levels. These twistings first appeared in its most generality in [8, 9], but only in [6] that explicit expressions for the modular matrices S and T appeared. For each cohomology class $[k] \in \mathrm{H}^3(G, \mathbb{T})$, they produce a modular data such that the (untwist) $k = 0$ model coincides with previously known model from the group G, i.e. the quantum double of G. The twisting was incorporated in the quantum-group picture in [9] and in the subfactor setting in [19], see also [11]. For all these reasons [6] the modular data that arise from the twist $[k]$ of a finite group is called the level k quantum G double modular data. The twisted primary fields are labelled by pairs $(a, \tilde{\pi})$ with a running in a set of conjugacy class representatives and $\tilde{\pi}$ are certain projective irreducible representations of the centralizer $C_G(a)$, see [6, (5.17)].

We work in particular models whose primary fields are simple currents, i.e. the quantum dimension $\dim(\lambda) = \frac{S_{0\lambda}}{S_{00}}$ of every primary field λ is 1, so due to the Verlinde formula the system ${}_N\mathcal{X}_N$ has a finite Abelian group $G = \{\lambda_g\}$ structure. For simple current modular data, the dual endomorphism of a braided subfactor $N \subset M$ is of the form $\theta_H = \bigoplus_{h\in H} \lambda_h$ for some subgroup H of G. The same θ_H may arise different (inner conjugate) subfactors and this is detected by the cohomology elements of $H^2(H, \mathbb{T})$, see [18]. Then we can compute the ${}_N\mathcal{X}_M$ system and thus its trace [15]:

$$\mathrm{Tr}(Z_{N\subset M}) = \frac{|G|}{|H|}. \tag{8}$$

For a given subgroup H the problem of when a θ_H gives rise to a Q-system, thus to a braided subfactor in the level k quantum G double data, has been addressed in [13, 17], however let us point out that its origin is Rehren's net setting [30]. In the level k quantum G double modular data, with G Abelian group, and T its modular matrix, θ_H can be endowed with a structure of a Q-system if and only if $T_{(a,l),(a,l)}^{N_{(a,l)}} = 1$ for all $(a, l) \in H$ where $N_{(a,l)}$ denotes the order of (a, l). Note that the simple currents are $G \times G$ as a set, see e.g. [6], and thus $H < G \times G$.

The plan of the rest of the paper is as follows. In section 2, we rewrite the quantum $\mathbb{Z}_n$ double modular data from any twist or level $[k]$, thus proving in proposition 2.1 that the modular data for the levels $[k]$ and $[-k]$ are the same up to a concrete permutation of the simple currents and complex conjugation of the modular S, T-matrices. Then in proposition 2.3 we produce some lower bounds for the total number of modular invariants of the level k quantum $\mathbb{Z}_n$ double modular data (implying that in the untwist $k = 0$ case, this number increases with the cardinal of the group, in contrast with level $k \neq 0$ where the number of modular invariants are few as the n prime cases show). Finally, in section 3 we complete the study for the quantum $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ models at all levels.

2. Twisted modular invariants from cyclic groups

Here, we are interested in analysing the twisted modular data that arise from the quantum double of cyclic groups $G = \mathbb{Z}_n$. First note that $\mathrm{H}^3(G, \mathbb{T}) = \mathbb{Z}_n$, thus we have n twists or levels. Take $[k] \in \mathrm{H}^3(G, \mathbb{T})$. In the following we also denote $[k]$ simply by k and the inverse of $[k]$, as an element of $\mathrm{H}^3(G, \mathbb{T})$, will be denoted by $-k$ or $n-k$. The primary fields of the quantum G double at level k equals $\mathbb{Z}_n \times \mathbb{Z}_n$ as a set, whose elements we will denote by $\lambda_{a,l}$ or simply by (a, l) for all $a, l \in G$. The modular data $S^{(k)}, T^{(k)}$ are derived explicitly in [6, equations (6.2) and (6.3)]:

$$S^{(k)}_{(a,l),(b,r)} = \frac{1}{n}\exp\left(-2\pi\mathrm{i}\frac{2kab + n(ar+bl)}{n^2}\right),$$
$$T^{(k)}_{(a,l),(a,l)} = \exp\left(2\pi\mathrm{i}\frac{ka^2 + nal}{n^2}\right). \tag{9}$$

Whenever no confusion arises, we will just write S and T instead of $S^{(k)}$ and $T^{(k)}$. The vacuum is $(0, 0)$ and for any level k it is obvious that every primary field (a, l) has quantum dimension $\frac{S_{(a,l),(0,0)}}{S_{(0,0),(0,0)}} = 1$, thus they are all simple currents and hence they form an Abelian group. The multiplication law of this Abelian group is, by the Verlinde formula [31],

$$(a, l)\cdot(b, r) = (a + b, l + r + 2k(a + b - \langle a + b\rangle)/n) \mod n\mathbb{Z}^2 \tag{10}$$

where $\langle a + b\rangle$ denotes $a + b$ mod n, see [6]. Clearly that for any $a \neq 0$ and l,

$$(0, l)^{-1} = (0, -l), \qquad (a, l)^{-1} = (n - a, -l - 2k). \tag{11}$$

If we set $f = \mathrm{GCD}(2k, n)$ with s, s' so that $fs = n$ and $fs' = 2k$ and GCD stands for the greatest common divisor, then the Abelian group structure of the primary fields is

$$\chi_{(G,k)} := \mathbb{Z}_f \times \mathbb{Z}_{n^2/f} \tag{12}$$

with (s, s') a generator of $\mathbb{Z}_f$ and $(1, 0)$ a generator of $\mathbb{Z}_{n^2/f}$. Note that in both the untwisted $k = 0$ case and $k = n/2$ if n is even (i.e. when $[k] = [-k]$), we obtain $f = n$ and therefore $\chi_{(G,k)} \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ as groups.

Of course $[0] = [-0]$, hence in the following we concentrate at the other levels.

Proposition 2.1. *Up to permutations of simple currents and complex conjugation, the modular data for $[k]$ and its inverse twist $[-k]$ coincide, namely,*

$$\overline{S^{(n-k)}_{(a,l),(b,r)}} = S^{(k)}_{(a,-a-l),(b,-b-r)} \qquad \text{and} \qquad \overline{T^{(n-k)}_{(a,l),(b,r)}} = T^{(k)}_{(a,-a-l),(b,-b-r)}.$$

Proof. By (9), we have

$$\begin{aligned}\overline{S^{(n-k)}_{(a,l),(b,r)}} &= \frac{1}{n}\exp\left(-2\pi\mathrm{i}\frac{-2(n-k)ab - n(ar+bl)}{n^2}\right)\\ &= \frac{1}{n}\exp\left(-2\pi\mathrm{i}\frac{2kab + n(a(-b-r) + b(-a-l))}{n^2}\right)\\ &= S^{(k)}_{(a,-a-l),(b,-b-r)}.\end{aligned}$$

Using again (9) we similarly obtain the identity for the T matrices:

$$\begin{aligned}\overline{T^{(n-k)}_{(a,l),(a,l)}} &= \exp\left(2\pi\mathrm{i}\frac{-na^2 + ka^2 - nal}{n^2}\right)\\ &= \exp\left(2\pi\mathrm{i}\frac{ka^2 + na(-a-l)}{n^2}\right) = T^{(k)}_{(a,-a-l),(a,-a-l)}.\end{aligned}$$

□

For $k \neq 0$, the map $\sigma : \chi_{(G,-k)} \to \chi_{(G,k)}$ sending (a, l) to $(a, -a - l)$ is easily seen to be an isomorphism of groups such that $\sigma^2 = \mathrm{id}$.

Table 1. Number of modular invariants of quantum $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ at all levels.

	Level 0	Level 1	Level 2	Level 3
$\mathbb{Z}_2$	6	2	–	–
$\mathbb{Z}_3$	9	3	3	–
$\mathbb{Z}_4$	22	4	10	4

Corollary 2.2. *The fusion rules and moreover the number of modular invariants for modular for the levels k and n − k coincide. Moreover, if* $Z = \sum Z_{(a,l)(b,r)} \chi_{al} \chi^*_{br}$ *is a modular invariant for the modular data at level k, then* $Z_\sigma := \sum Z_{(a,-a-l)(b,-b-r)} \chi_{a,-a-l} \chi^*_{b,-b-r}$ *is a modular invariant for the level* $n - k$.

Proof. The result follows from proposition 2.1 and the fact that such a matrix Z commutes with S and T iff Z commutes with the complex conjugation of S and T. □

We similarly yield a modular invariant Z_σ at level k from a given modular invariant Z at level $n - k$, and moreover $(Z_\sigma)_\sigma = Z$ since $\sigma^2 = \mathrm{id}$.

2.1. Some modular invariants

It has been proved in [6] that there are at least five modular invariants in the quantum double of any finite group at level zero. Next result is an improvement of that result with some partial results for any level.

Proposition 2.3. *Let us consider the quantum* $\mathbb{Z}_n$ *double level k modular data. Then,*

(i) the number of level 0 (sufferable) modular invariants is greater than $n + 3$.
If $n \geqslant 6$ *is even then the number of modular invariants is greater than* $n + 10$;
(ii) if $GCD(2k, n) = 1$ *for a level* $k \neq 0$, *the number of modular invariants is precisely the number of divisors of* n^2. *Moreover if n is an odd prime, then* $Z_1 = \mathrm{id}$, $Z_2 = C$ *and* $Z_3 = \sum \chi_{0i} \chi^*_{0i}$ *are all the (sufferable) quantum* $\mathbb{Z}_n$ *double modular invariants at level k.*

Proof. For level 0, the system ${}_N\mathcal{X}_N$ is isomorphic to the Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. Since $T^n_{10,10} = T^n_{01,01} = 1$ we conclude that $\theta_{\mathbb{Z}_n \times 0}$ and $\theta_{0 \times \mathbb{Z}_n}$ are dual endomorphisms and therefore $\theta_{\mathbb{Z}_n \times \mathbb{Z}_n} = \theta_{\mathbb{Z}_n \times 0} \cdot \theta_{0 \times \mathbb{Z}_n}$ is also a dual endomorphism by [13]. Hence θ_H is a dual endomorphism, for any subgroup $H < \mathbb{Z}_n \times \mathbb{Z}_n$.

Next let us consider the following list of subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$:

$$H_s = \langle (1, s) \rangle, \qquad s = 0, 1, \ldots, n - 1.$$

Clearly, $H_1 = \{(h, h): h \in \mathbb{Z}_n\}$ and $H_0 = \mathbb{Z}_n \times 0$. Together with $0 \times \mathbb{Z}_n$ we get in this way $n+1$ different copies of $\mathbb{Z}_n$ inside $\mathbb{Z}_n \times \mathbb{Z}_n$. Of course, we have the extra two trivial subgroups. By [15, lemma 3.2] or [29, section 3], two different subgroups in this list of $n + 3$ subgroups are attached to different modular invariants (even if they are isomorphic as abstract groups). If n is even then we have a subgroup $\mathbb{Z}_2 = \{0, n/2\}$. For $n \neq 2, 4$ we get new subgroups

$$\mathbb{Z}_2 \times 0, 0 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}_2, \quad \{(0, 0), (n/2, n/2)\}, \quad H_C := \{(2a, 2r)\}.$$

Note that $|H_C| = n^2/4$. So we get seven new subgroups in the even case and therefore $n + 10$ modular invariants.

We proceed now with the proof of (ii). If $\mathrm{GCD}(2k, n) = 1$, then the system ${}_N\mathcal{X}_N \simeq \mathbb{Z}_{n^2}$ as groups, see (12), with $(1, 0)$ its generator. Since $T^{n^2}_{10,10} = 1$ we conclude that $\theta_{\mathbb{Z}_{n^2}}$ is a dual endomorphism and so are all their subgroups. Every divisor of n^2 gives rise to a subgroup

Table 2. Quantum $\mathbb{Z}_2$ modular invariants: levels 0 and 1.

Modular invariant	Trace	θ_H
Level 0		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	4	0×0
$Z_2 = \sum \chi_{ij}\chi_{ji}^*$	2	$\Delta(\mathbb{Z}_2)$
$Z_3 = \sum \chi_{0i}\chi_{0j}^*$	2	$0 \times \mathbb{Z}_2$
$Z_4 = \sum \chi_{i0}\chi_{j0}^*$	2	$\mathbb{Z}_2 \times 0$
$Z_5 = \sum \chi_{0j}\chi_{i0}^*$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_6 = \sum \chi_{i0}\chi_{0j}^*$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Level 1		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	4	0×0
$Z_2 = \sum \chi_{0i}\chi_{0j}^*$	2	$0 \times \mathbb{Z}_2$

Table 3. Quantum $\mathbb{Z}_3$ modular invariants.

Modular invariants	Trace	θ_H
Level 0		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	9	0×0
$Z_2 = \sum \chi_{ij}\chi_{ji}^*$	3	$\{(0,0),(1,2),(2,1)\}$
$Z_3 = \sum \chi_{ij}\chi_{-j-i}^*$	3	$\Delta(\mathbb{Z}_3)$
$Z_4 = \sum \chi_{i0}\chi_{j0}^*$	3	$\mathbb{Z}_3 \times 0$
$Z_5 = \sum \chi_{0i}\chi_{0j}^*$	3	$0 \times \mathbb{Z}_3$
$Z_6 = C = \sum \chi_{ij}\chi_{-i-j}^*$	1	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$Z_7 = \sum \chi_{i0}\chi_{0j}^*$	1	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$Z_8 = \sum \chi_{0j}\chi_{i0}^*$	1	$\mathbb{Z}_3 \times \mathbb{Z}_3$
Levels 1 and 2		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	9	0×0
$Z_2 = \sum \chi_{0i}\chi_{0i}^*$	3	$0 \times \mathbb{Z}_3$
$Z_3 = C = \sum \chi_{0j}\chi_{0,-j}^* + \sum_{i\neq 0}\chi_{ij}\chi_{3-i,-j-2}^*$	1	$\mathbb{Z}_9$

of $\mathbb{Z}_{n^2}$, so there the number of modular invariants is at least the number of divisors of n^2, [15, lemma 3.2]. On the other hand, every such subgroup K of $\mathbb{Z}_{n^2}$ is itself cyclic and therefore the second cohomology $\mathrm{H}^2(K,\mathbb{T})$ is trivial. Hence, every subgroup has only one Q-system structure, see [13, 18]. Therefore, every subgroup of $\mathbb{Z}_{n^2}$ is attached to exactly one modular invariant.

If n is prime then we get exactly the three obvious divisors and so three sufferable modular invariants. □

In particular, in the untwist $k = 0$ case, the number of modular invariants increases as the cardinal of the group increases. For n prime, the quantum $\mathbb{Z}_n$ double level 0 (sufferable) modular invariants are fully treated in [15].

3. Examples: quantum $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ at all levels

Here, we work out all the twisted models arising from the quantum G double where $G = \mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_4$ at all levels $[k] \in \mathrm{H}^3(G,\mathbb{T})$. The quantum $\mathbb{Z}_2$ level 0 was done in [3, 13], the quantum $\mathbb{Z}_3$ all levels were done in [13, section 5], while the quantum $\mathbb{Z}_4$ level 0 was done in [29, section 3]. Thus in this section we study the remaining cases: first

Table 4. Quantum $\mathbb{Z}_4$ level 0 modular invariants.

Modular invariants	Trace	Canonical θ_H
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	16	0×0
$Z_2 = C = \sum \chi_{ij}\chi_{-i-j}^*$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_3 = \sum \chi_{ij}\chi_{ji}^*$	4	$\{(0,0),(1,3),(3,1),(2,2)\}$
$Z_4 = \sum \chi_{ij}\chi_{-j-i}^*$	4	$\Delta(\mathbb{Z}_4)$
$Z_5 = \sum \chi_{0i}\chi_{0j}^*$	4	$0 \times \mathbb{Z}_4$
$Z_6 = \sum \chi_{i0}\chi_{j0}^*$	4	$\mathbb{Z}_4 \times 0$
$Z_7 = \lvert\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22}\rvert^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_8 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{21}+\chi_{23}\rvert^2 + (\chi_{01}+\chi_{03})(\chi_{20}+\chi_{22})^* + (\chi_{20}+\chi_{22})(\chi_{01}+\chi_{03})^*$	4	$\{(0,0),(0,2),(2,1),(2,3)\}$
$Z_9 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{01}+\chi_{03}\rvert^2 + \lvert\chi_{20}+\chi_{22}\rvert^2 + \lvert\chi_{21}+\chi_{23}\rvert^2$	8	$0 \times \mathbb{Z}_2$
$Z_{11} = \lvert\chi_{00}+\chi_{22}\rvert^2 + \lvert\chi_{02}+\chi_{20}\rvert^2 + \lvert\chi_{11}+\chi_{33}\rvert^2 + \lvert\chi_{13}+\chi_{31}\rvert^2$	8	$\Delta(\mathbb{Z}_2)$
$Z_{22} = \lvert\chi_{00}+\chi_{20}\rvert^2 + \lvert\chi_{12}+\chi_{32}\rvert^2 + (\chi_{02}+\chi_{22})(\chi_{10}+\chi_{30})^* + (\chi_{10}+\chi_{30})(\chi_{02}+\chi_{22})^*$	4	$\{(0,0),(2,0),(1,2),(3,2)\}$
$Z_{33} = \lvert\chi_{00}+\chi_{20}\rvert^2 + \lvert\chi_{02}+\chi_{22}\rvert^2 + \lvert\chi_{10}+\chi_{30}\rvert^2 + \lvert\chi_{12}+\chi_{33}\rvert^2$	8	$\mathbb{Z}_2 \times 0$
$Z_{56} = \sum \chi_{0i}\chi_{j0}^*$	1	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$Z_{65} = \sum \chi_{i0}\chi_{0j}^*$	1	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$Z_{57} = \sum \chi_{0i}(\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})^*$	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$Z_{75} = \sum(\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})\chi_{0i}^*$	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$Z_{67} = \sum \chi_{i0}(\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})^*$	2	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$Z_{76} = \sum(\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})\chi_{i0}^*$	2	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$Z_{(2)7} = (\chi_{00}+\chi_{20})(\chi_{00}+\chi_{02})^* + (\chi_{02}+\chi_{22})(\chi_{01}+\chi_{03})^* + (\chi_{10}+\chi_{30})(\chi_{20}+\chi_{22})^* + (\chi_{12}+\chi_{32})(\chi_{21}+\chi_{23})^*$	1	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$Z_{(2)7} = (\chi_{00}+\chi_{02})(\chi_{00}+\chi_{20})^* + (\chi_{01}+\chi_{03})(\chi_{02}+\chi_{22})^* + (\chi_{20}+\chi_{22})(\chi_{10}+\chi_{30})^* + (\chi_{21}+\chi_{23})(\chi_{12}+\chi_{32})^*$	1	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$Z_{(2)8} = (\chi_{00}+\chi_{20})(\chi_{00}+\chi_{02})^* + (\chi_{10}+\chi_{30})(\chi_{01}+\chi_{03})^* + (\chi_{02}+\chi_{22})(\chi_{20}+\chi_{22})^* + (\chi_{12}+\chi_{32})(\chi_{21}+\chi_{23})^*$	2	$\{(0,0),(0,2),(1,1),(1,3),(2,0),(2,2),(3,1),(3,3)\}$
$Z_{8(2)} = (\chi_{00}+\chi_{02})(\chi_{00}+\chi_{20})^* + (\chi_{01}+\chi_{03})(\chi_{10}+\chi_{30})^* + (\chi_{20}+\chi_{22})(\chi_{02}+\chi_{22})^* + (\chi_{21}+\chi_{23})(\chi_{12}+\chi_{32})^*$	2	$\{(0,0),(0,2),(1,1),(1,3),(2,0),(2,2),(3,1),(3,3)\}$

we numerically compute the modular invariants for which we use a computer program and [3, (1.6)] which says that in any modular data $Z_{\lambda\mu} \leqslant \dim(\lambda)\dim(\mu)$ thus in our simple currents cases $Z_{\lambda\mu} \in \{0, 1\}$. The number of modular invariants are as in table 1. For completeness, we write all the modular invariants together with their traces and the canonical endomorphism θ_H that produces every modular invariant (see tables 2–5). In order to save space, instead of using the matrix $Z = [Z_{ij,ab}]$ we use the partition function notation $Z = \sum Z_{ij,ab}\chi_{ij}\chi_{ab}^*$, where χ's here are regarded as symbols. For $K < G$ we denote by $\Delta(K) = \{(k,k) : k \in K\}$ the diagonal copy of K in $G \times G$.

For the level 1 model, we have two modular invariants, see the RHS of table 2. Note that in this case $H = \{(0,0),(0,2)\}$ gives rise to a canonical endomorphism θ_H by [13, lemma 3.8] since $T^2_{02,02} = 1$. This θ_H produces the modular invariant Z_2 as the trace of the modular invariant associated with θ_H has to be 2 by (7). Also remark that the system ${}_N\mathcal{X}_N$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as a group. Nevertheless, the subgroup $K = \{(0,0),(2,0))\}$ does not give rise to a canonical endomorphism θ_K, again by [13, lemma 3.8] since in this case $T^2_{02,02} \neq 1$ (neither is the full group $\mathbb{Z}_2 \times \mathbb{Z}_2$).

Table 5. Quantum $\mathbb{Z}_4$ modular invariants: levels 1, 2 and 3.

Modular invariant	Trace	θ_H
Levels 1 and 3		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	16	0×0
$Z_2 = \sum \chi_{0i}\chi_{0-i}^* + \sum_{i\neq 0} \chi_{ij}\chi_{-i,-j-2}^*$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_3 = \sum \chi_{0i}\chi_{0j}^*$	4	$0 \times \mathbb{Z}_4$
$Z_4 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{01}+\chi_{03}\rvert^2 + \lvert\chi_{20}+\chi_{22}\rvert^2 + \lvert\chi_{21}+\chi_{23}\rvert^2$	8	$0 \times \mathbb{Z}_2$
Level 2		
$Z_1 = \sum \chi_{ij}\chi_{ij}^*$	16	0×0
$Z_2 = \sum \chi_{ij}\chi_{-i-j}^*$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_3 = \sum_{i+j \text{ even}} \chi_{ij}\chi_{ij}^* + \sum_{2i+j \text{ odd}} \chi_{ij}\chi_{i+2,j}^*$	8	$\mathbb{Z}_2 \times 0$
$Z_4 = \sum_{j \text{ even}} \chi_{ij}\chi_{ij}^* + \sum_{2i+j \text{ odd}} \chi_{ij}\chi_{i+2,j+2}^*$	8	$\Delta(\mathbb{Z}_2)$
$Z_5 = \sum \chi_{0i}\chi_{0j}^*$	4	$0 \times \mathbb{Z}_4$
$Z_6 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{20}+\chi_{22}\rvert^2$ $(\chi_{01}+\chi_{03})(\chi_{11}+\chi_{13})^* + (\chi_{11}+\chi_{12})(\chi_{01}+\chi_{03})^*$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Z_7 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{21}+\chi_{23}\rvert^2$ $(\chi_{00}+\chi_{02})(\chi_{21}+\chi_{22})^* + (\chi_{21}+\chi_{22})(\chi_{00}+\chi_{02})^*$	4	$\{(0,0),(0,2),(2,1),(2,3)\}$
$Z_8 = \lvert\chi_{00}+\chi_{02}\rvert^2 + \lvert\chi_{01}+\chi_{03}\rvert^2 + \lvert\chi_{20}+\chi_{22}\rvert^2 + \lvert\chi_{21}+\chi_{23}\rvert^2$	8	$0 \times \mathbb{Z}_2$
$Z_{57} = \sum \chi_{0i}(\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})^*$	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$Z_{75} = \sum (\chi_{00}+\chi_{02}+\chi_{20}+\chi_{22})\chi_{0i}^*$	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$

The levels 0, 1 and 2 of the quantum $\mathbb{Z}_3$ double were studied in [13, section 5.1] where it was also noted that the level 1 data coincide with the WZW $SU(9)$ level 1 data. Thanks to proposition 2.1 we now know that the modular data for the level 2 have to be the complex conjugation to the modular data of the level 1 (with relabelling of the simple currents). Note that for the levels $k = 1$ and $k = 2$ we have $\mathrm{GCD}(2k, n) = 1$, thus in both cases the fusion rules of the system ${}_N\mathcal{X}_N$ is $\mathbb{Z}_9$, see (12).

Theorem 3.1. *All the modular invariants of the quantum $\mathbb{Z}_4$ double models at every level are realized by subfactors.*

Proof. By [29] we know that all the 22 level 0 quantum $\mathbb{Z}_4$ modular invariants are realized by subfactors, see table 4.

Now we study the level 1 data. There are precisely four modular invariants, see table 5. The system ${}_N\mathcal{X}_N$ has the structure of the group $\mathbb{Z}_2 \times \mathbb{Z}_8$, as $\mathrm{GCD}(4, 2) = 2$, see (12). Let us consider the following list of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$:

$$H_2 = \{(0,0),(0,2),(2,0),(2,2)\}, \qquad H_3 = 0 \times \mathbb{Z}_4, \qquad H_4 = \{(0,0),(0,2)\}.$$

Note that by the fusion rules, see (12) with $k = 1$ and $n = 4$, the group H_2 is isomorphic to $\mathbb{Z}_4$ with $(2, 0)$ being a generator and H_4 a copy of $\mathbb{Z}_2$. Since $T^4_{20,20} = 1 = T^4_{01,01}$, in [17] it is implied that θ_{H_2} and θ_{H_3} are canonical endomorphisms. Hence θ_{H_4} is also a canonical endomorphism since H_4 is a subgroup of H_2.

On the other hand, the RHS of (7) for the modular invariants Z_2, Z_3 and Z_4 is $4[\theta_{H_2}]$, $4[\theta_{H_3}]$ and $8[\theta_{H_4}]$, respectively. Therefore, H_2, H_3 and H_4 produce the modular invariants Z_2, Z_3 and Z_4, respectively. Thanks to proposition 2.1, level 3 model reduces to that of level 1.

Finally, we study the level 2 model. There are ten modular invariants as written in table 5. The system ${}_N\mathcal{X}_N$ has the structure of the group $\mathbb{Z}_4 \times \mathbb{Z}_4$, as $\mathrm{GCD}(4, 4) = 4$. Let us consider the following list of subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$:

$$H_3 = \{(0,0),(2,0)\}, \qquad H_4 = \{(0,0),(2,2)\}, \qquad H_5 = 0 \times \mathbb{Z}_4,$$
$$H_7 = \{(0,0),(0,2),(2,1),(2,2)\}, \qquad H_8 = \{(0,0),(0,2)\}.$$

Since (21) is a generator of H_7, a copy of $\mathbb{Z}_4$, and $T^4_{21,21} = 1$ we conclude that θ_{H_7} is a canonical endomorphism (similarly with H_5). The others are subgroups of H_5 or H_7, hence $\theta_{H_3}, \theta_{H_4}$ and θ_{H_8} are canonical endomorphisms. By computing the RHS of (7) for all the modular invariants we conclude that every canonical endomorphism $\theta_{H_3}, \theta_{H_4}, \theta_{H_5}, \theta_{H_7}$ and θ_{H_8} appears only for Z_3, Z_4, Z_5, Z_7 and Z_8. Therefore, θ_{H_i} produces Z_i, with $i = 3, 4, 5, 7, 8$. Since we have $Z_3 Z_4 = Z_2$, $Z_3 Z_8 = Z_6$ and $Z_3 Z_5 = Z_{75}$ we conclude that the other modular invariants are also sufferable (using [13, theorem 3.6]). □

Acknowledgment

The author is partially supported by the Fundação para a Ciência e a Tecnologia through the Program POCI 2010/FEDER.

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